

High-order Cheeger's inequality on domain

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Abstract

We study the relationship of higher order variational eigenvalues of p -Laplacian and the higher order Cheeger constants. The asymptotic behavior of the k -th Cheeger constant is investigated. Using methods developed in [2], we obtain the high-order Cheeger's inequality of p -Laplacian on domain $h_k^p(\Omega) \leq C\lambda_k(p, \Omega)$.

Keywords: High order Cheeger's inequality; eigenvalue problem; p -Laplacian

1 INTRODUCTION.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain. The minimax of the so-called Rayleigh quotient

$$\lambda_k(p, \Omega) = \inf_{A \in \Gamma_{k,p}} \max_{u \in A} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} |u|^p dx}, \quad (1 < p < \infty), \quad (1.1)$$

leads to a nonlinear eigenvalue problem, where

$$\Gamma_{p,k} = \{A \in W_0^{1,p}(\Omega) \setminus \{0\} \mid A \cap \{\|u\|_p = 1\} \text{ is compact, Asymmetric, } \gamma(A) \geq k\}.$$

The corresponding Euler-Lagrange equation is

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda |u|^{p-2} u, \quad (1.2)$$

with Dirichlet boundary condition. This eigenvalue problem has been extensively studied in the literature. When $p = 2$, it is the familiar linear Laplacian equation

$$\Delta u + \lambda u = 0.$$

The solution of this Laplacian equation describes the shape of an eigenvibration, of frequency $\sqrt{\lambda}$, of homogeneous membrane stretched in the frame Ω . It is well-known that the spectrum of Laplacian equation is discrete and all eigenfunctions form an orthonormal basis for $L^2(\Omega)$ space. For general $1 < p < \infty$, the first eigenvalue $\lambda_1(p, \Omega)$ of p -Laplacian $-\Delta_p$ is simple and isolated.

The second eigenvalue $\lambda_2(p, \Omega)$ is well-defined and has a “variational characterization”, see [20]. It has exactly 2 nodal domains, c.f.[14]. However, we know little about the higher eigenvalues and eigenfunctions of the p -Laplacian when $p \neq 2$. It is unknown whether the variational eigenvalues (1.1) can exhaust the spectrum of equation (1.2). In this paper, we only discuss the variational eigenvalues (1.1). For (1.1), there are asymptotic estimates, c.f.[17] and [18]. [21], [22], and [23] discuss the p -Laplacian eigenvalue problem as $p \rightarrow \infty$ and $p \rightarrow 1$.

The Cheeger’s constant which was first studied by J.Cheeger in [9] is defined by

$$h(\Omega) := \inf_{D \subseteq \Omega} \frac{|\partial D|}{|D|}, \quad (1.3)$$

with D varying over all smooth subdomains of Ω whose boundary ∂D does not touch $\partial\Omega$ and with $|\partial D|$ and $|D|$ denoting $(n-1)$ and n -dimensional Lebesgue measure of ∂D and D . We call a set $C \subseteq \overline{\Omega}$ Cheeger set of Ω , if $\frac{|\partial C|}{|C|} = h(\Omega)$. For more about the uniqueness and regularity, we refer to [11]. Cheeger sets are of significant importance in the modelling of landslides, see [24],[25], or in fracture mechanics, see [26].

The classical Cheeger’s inequality is about the first eigenvalue of Laplacian and the Cheeger constant(c.f.[3])

$$\lambda_1(2, \Omega) \geq \left(\frac{h(\Omega)}{2} \right)^2 \quad \text{i.e.} \quad h(\Omega) \leq 2\sqrt{\lambda_1(2, \Omega)},$$

which was extent to the p -Laplacian in [12]:

$$\lambda_1(p, \Omega) \geq \left(\frac{h(\Omega)}{p} \right)^p.$$

When $p = 1$, the first eigenvalue of 1-Laplacian is defined by

$$\lambda_1(1, \Omega) := \min_{0 \neq u \in BV(\Omega)} \frac{\int_{\Omega} |Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{n-1}}{\int_{\Omega} |u| dx}, \quad (1.4)$$

where $BV(\Omega)$ denotes the space of functions of bounded variation in Ω . From [3], $\lambda_1(1, \Omega) = h(\Omega)$. And, problem (1.3) and problem (1.4) are equivalent in the following sense: a function $u \in BV(\Omega)$ is a minimum of (1.4) if and only if almost every level set is a Cheeger set. An important difference between $\lambda_1(p, \Omega)$ and $h_k(\Omega)$ is that the first eigenfunction of p -Laplacian is unique while the uniqueness of Cheeger set depends on the topology of the domain. For counterexamples, see [4, Remark 3.13]. For more results about the eigenvalues of 1-Laplacian, we refer to [6] and [7].

As to the more general Lipschitz domain, we need the following definition of perimeter:

$$P_{\Omega}(E) := \sup \left\{ \int_E \operatorname{div} \phi dx \mid \phi \in C_c^1(\Omega, \mathbb{R}^n), |\phi| \leq 1, \operatorname{div} \phi \in L^{\infty}(\Omega) \right\}.$$

For convenience, we denote $|\partial E| := P_{\Omega}(E)$. The higher order Cheeger’s constant is defined by

$$h_k(\Omega) := \inf \{ \lambda \in \mathbb{R}^+ \mid \exists E_1, E_2, \dots, E_k \subseteq \Omega, E_i \cap E_j = \emptyset, i \neq j, \max_{1,2,\dots,k} \frac{|\partial E_i|}{|E_i|} \leq \lambda \};$$

if $|E| = 0$, we set $\frac{|\partial E|}{|E|} = +\infty$. An equivalent characterization of the higher order Cheeger constant is (see[4])

$$h_k(\Omega) := \inf_{\mathfrak{D}_k} \max_{i=1,2,\dots,k} h(E_i),$$

where \mathfrak{D}_k are the set of all partitions of Ω with k subsets. We set $h_1(\Omega) := h(\Omega)$. Obviously, if $R \subseteq \Omega$, then $h_k(\Omega) \leq h_k(R)$.

For the high-order Cheeger constants, there is a conjecture:

$$\lambda_k(p, \Omega) \geq \left(\frac{h_k(\Omega)}{p} \right)^p. \quad \forall 1 \leq k < +\infty, 1 < p < +\infty. \quad (1.5)$$

From [14, Theorem 3.3], the second variational eigenfunction of $-\Delta_p$ has exactly two nodal domains, see also [20]. It follows that (1.5) is hold for $k = 1, 2$. We refer to [4, Theorem 5.4] for more details. However, by Courant's nodal domain theorem, for other variational eigenfunctions, it is not necessary to have exactly k nodal domains. Therefore, the inequality (1.5) on domain is still an open problem for $k > 2$.

In this paper, we will get an asymptotic estimate for $h_k(\Omega)$ and establish high-order Cheeger's inequality for general k , and discuss the reversed inequality. To deal with the high-order Cheeger's inequality, we should give some restriction on domain.

Definition 1.1. *If there exists n -dimensional rectangle $R \subset \Omega$ and c_1, c_2 independent of Ω , such that $c_1|R| \leq |\Omega| \leq c_2|R|$, we say R the comparable inscribed rectangle of Ω .*

In graph theory, when $p = 2$ the high-order Cheeger inequality was proved in [1], and was improved in [2]. In [1], using orthogonality of the eigenfunctions of Laplacian in l_2 and a random partitioning, they got

$$\frac{\lambda_k}{2} \leq \rho_G(k) \leq O(k^2) \sqrt{\lambda_k},$$

where $\rho_G(k)$ is the k -way expansion constant, the analog of h_k . But, when it comes to the domain case, there is no such random partitioning. Therefore, we adapt the methods in [2] to get:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a comparable inscribed rectangle. For $1 < p < \infty$, we have the following asymptotic estimates:*

$$h_k(\Omega) \leq C k^{\frac{1}{n}} \left(\frac{\lambda_1(p, \Omega)}{h_1(\Omega)} \right)^{\frac{q}{p}}, \quad \forall 1 \leq k < +\infty \quad (1.6)$$

where C only depends on n, p , $\frac{1}{p} + \frac{1}{q} = 1$.

There are some lower bounds about the first eigenvalue of p -Laplacian, see [19]. There is lower bound by the h_k when Ω be a planar domain with finite connectivity k .

Theorem 1.3 ([8]). *Let (S, g) be a Riemannian surface, and let $D \subset S$ be a domain homeomorphic to a planar domain of finite connectivity k . Let F_k be the family of relatively compact subdomains of D with smooth boundary and with connectivity at most k . Let*

$$h_k(D) := \inf_{D' \in F_k} \frac{|\partial D'|}{|D'|},$$

where $|D'|$ is the area of D' and $|\partial D'|$ is the length of its boundary. Then,

$$\lambda_1(p, D) \geq \left(\frac{h_k(D)}{p} \right)^p.$$

Remark 1.4. The results of theorem 1.2 generalize the above theorem to more general cases.

As to the reversed inequality, if $\Omega \subset \mathbb{R}^n$ is convex, the following lower bound (the Faber-Krahn inequality) for $h_k(\Omega)$ was proved in [4]:

$$h_k(\Omega) \geq n \left(\frac{k\omega_n}{|\Omega|} \right)^{\frac{1}{n}}, \quad \forall k = 1, 2, \dots, \quad (1.7)$$

where ω_n is the volume of the unit ball in \mathbb{R}^n . Therefore

$$0 < h_1(\Omega) \leq h_2(\Omega) \leq \dots \leq h_k(\Omega) \rightarrow +\infty, \text{ as } k \rightarrow \infty. \quad (1.8)$$

However, for general domain, inequalities (1.7) and (1.8) are not true at all for $p > 1$. In fact, there are counter-examples in [15] and [16] to show that there exist domains such that $h_k(\Omega) \leq c$, where c depends only on n . Meanwhile, $\lambda_k(p, \Omega) \rightarrow +\infty$. Therefore, the reversed inequality of (1.6) is not hold for general domain.

Let's consider the convex domain. By the John ellipsoid theorem (c.f.[27, theorem 1.8.2]) and the definition of comparable inscribed rectangle, there exists comparable inscribed rectangle R for convex Ω , such that $c_1|R| \leq |\Omega| \leq c_2|R|$, where c_1, c_2 depend only on n .

On the other hand, according to [17] and [18], for $1 < p < +\infty$, there exist C_1, C_2 depending only on p, n , such that

$$C_1 \left(\frac{k}{|\Omega|} \right)^{\frac{1}{n}} \leq \lambda_k^{\frac{1}{p}}(p, \Omega) \leq C_2 \left(\frac{k}{|\Omega|} \right)^{\frac{1}{n}}, \quad \forall k \in \mathbb{N}. \quad (1.9)$$

Therefore, if the domain is a bounded convex domain, combining Theorem 1.2, (1.7) and (1.9), the following inequality holds.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, then there exist C_1, C_2 depending only on n , such that*

$$C_1 \left(\frac{k}{|\Omega|} \right)^{\frac{1}{n}} \leq h_k(\Omega) \leq C_2 \left(\frac{k}{|\Omega|} \right)^{\frac{1}{n}}, \quad \forall k \in \mathbb{N}.$$

By the two theorems above, we get bilateral estimate of $h_k(\Omega)$ with respect to $\lambda_k(p, \Omega)$.

Corollary 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Then, for $1 < p < \infty$, there exist C_1, C_2 depending only on p, n , such that*

$$C_1 \lambda_k(p, \Omega) \leq h_k^p(\Omega) \leq C_2 \lambda_k(p, \Omega).$$

Remark 1.7. From [4], when $\Omega \subset \mathbb{R}^n$ be a lipschitz domain, there is

$$\limsup_{p \rightarrow 1} \lambda_k(p, \Omega) \leq h_k(\Omega).$$

This paper is arranged as follows: In section 2, we get some variants of Cheeger's inequalities. Section 3 is devoted to prove Theorem 1.2 and Theorem 1.5.

2 SOME VARIANTS OF CHEEGER'S INEQUALITIES

In this section, we will give several variants of Cheeger's inequalities. For a subset $S \subseteq \Omega$, define $\phi(S) = \frac{|\partial S|}{|S|}$. The Rayleigh quotient of ψ is defined by $\mathcal{R}(\psi) := \frac{\int_{\Omega} |\nabla \psi|^p dx}{\int_{\Omega} |\psi|^p dx}$. We define the support of ψ , $Supp(\psi) = \{x \in \Omega | \psi(x) \neq 0\}$. If $Supp(f) \cap Supp(g) = \emptyset$, we say f and g are disjointly supported. Let $\Omega_{\psi}(t) := \{x \in \Omega | \psi(x) \geq t\}$ be the level set of ψ . For an interval $I = [t_1, t_2] \subseteq \mathbb{R}$. $|I| = |t_2 - t_1|$ denote the length of I . For any function ψ , $\Omega_{\psi}(I) := \{x \in \Omega | \psi(x) \in I\}$, $\phi(\psi) := \min_{t \in \mathbb{R}} \phi(\Omega_{\psi}(t))$. $t_{opt} := \min\{t \in \mathbb{R} | \phi(\Omega_{\psi}(t)) = \phi(\psi)\}$.

Lemma 2.1. *For any $\psi \in W_0^{1,p}(\Omega)$, there exist a subset $S \subseteq Supp\psi$, such that $\phi(S) \leq p(\mathcal{R}(\psi))^{\frac{1}{p}}$.*

The proof can be found in the appendix of [11], we write it here for the reader's convenience.

Proof. Note that $|\nabla|\psi|| \leq |\nabla\psi|$. We only need to show the conclusion for $\psi \geq 0$. Suppose first that $\omega \in C_0^{\infty}(\Omega)$. Then by the coarea formula and by Cavalieri's principle

$$\begin{aligned} \int_{\Omega} |\nabla \omega| dx &= \int_{-\infty}^{\infty} |\partial \Omega_{\omega}(t)| dt = \int_{-\infty}^{\infty} \frac{|\partial \Omega_{\omega}(t)|}{|\Omega_{\omega}(t)|} |\Omega_{\omega}(t)| dt \\ &\geq \inf \frac{|\partial \Omega_{\omega}(t)|}{|\Omega_{\omega}(t)|} \int_{-\infty}^{+\infty} |\Omega_{\omega}(t)| dt = \inf \frac{|\partial \Omega_{\omega}(t)|}{|\Omega_{\omega}(t)|} \int_{\Omega} |\omega| dx = \phi(\Omega_{\omega}(t_{opt})) \int_{\Omega} |\omega| dx \end{aligned} \quad (2.1)$$

Since $C_0^{\infty}(\Omega)$ is dense in $W_0^{1,1}(\Omega)$, the above inequality also holds for $\omega \in W_0^{1,1}(\Omega)$. Define $\Phi(\psi) = |\psi|^{p-1}\psi$. Then Hölder's inequality implies

$$\int_{\Omega} |\nabla \Phi(\psi)| dx = p \int_{\Omega} |\psi|^{p-1} |\nabla \psi| dx \leq p \|\psi\|_p^{p-1} \|\nabla \psi\|_p.$$

Meanwhile, (2.1) applies and

$$\int_{\Omega} |\nabla \Phi(\psi)| \geq \phi(\Omega_{\Phi}(t_{opt})) \int_{\Omega} |\psi|^p dx.$$

Therefore, there exist a subset $S =: \Omega_{\Phi}(t_{opt}) \subseteq Supp\psi$, such that

$$\phi(S) \leq \frac{\int_{\Omega} |\nabla \omega| dx}{\int_{\Omega} |\omega| dx} \leq \frac{p \|\psi\|_p^{p-1} \|\nabla \psi\|_p}{\int_{\Omega} |\psi|^p dx} = p \frac{\|\nabla \psi\|_p}{\|\psi\|_p} = p(\mathcal{R}(\psi))^{\frac{1}{p}}.$$

□

Let $\mathcal{E}_f := \int_{\Omega} |\nabla f|^p dx$. Then $\mathcal{R}(f) = \frac{\mathcal{E}_f}{\|f\|_p^p}$. To use the classical Cheeger's inequality for truncated functions, we introduce $\mathcal{E}_f(I) := \int_{\Omega_f(I)} |\nabla f|^p dx$.

Lemma 2.2. For any function $f \in W_0^{1,p}(\Omega)$, and interval $I = [b, a]$ with $a > b \geq 0$, we have

$$\mathcal{E}_f(I) \geq \frac{(\phi(f)|\Omega_f(a)||I|)^p}{|\Omega_f(I)|^{\frac{p}{q}}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We first prove it for $f \in C_0^\infty(\Omega)$. By Coarea formula and Cavalieri's principle,

$$\begin{aligned} \int_{\Omega_f(I)} |\nabla f| dx &= \int_I |\partial \Omega_f(t)| dt = \int_I \frac{|\partial \Omega_f(t)|}{|\Omega_f(t)|} |\Omega_f(t)| dt \\ &\geq \phi(f) \int_I |\Omega_f(t)| dt \geq \phi(f) |\Omega_f(a)| |I|. \end{aligned}$$

The Hölder inequality gives

$$\left(\int_{\Omega_f(I)} |\nabla f| dx \right) \leq \left(\int_{\Omega_f(I)} |\nabla f|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega_f(I)} dx \right)^{\frac{1}{q}} = \left(\int_{\Omega_f(I)} |\nabla f|^p dx \right)^{\frac{1}{p}} |\Omega_f(I)|^{\frac{1}{q}}.$$

Combining above two inequalities, we get this Lemma for $f \in C_0^\infty(\Omega)$. Arguments as in the proof of Lemma 2.1 give this lemma for $f \in W_0^{1,p}(\Omega)$. \square

3 CONSTRUCTION OF SEPARATED FUNCTIONS

In this section, we will prove theorem 1.2 and theorem 1.5. We use the method developed in [2] for high-order Cheeger's inequality on graph. Our proof consists of three steps. First, we will deal with the case of n -dimensional rectangle $\Omega = (a_1, b_1) \times (a_2, b_2) \times \cdots (a_n, b_n) \subset \mathbb{R}^n$ with single variable changed. Second, we extend to the case of n -dimensional rectangle $\Omega = (a_1, b_1) \times (a_2, b_2) \times \cdots (a_n, b_n) \subset \mathbb{R}^n$ with multi-variables changed. Finally, we discuss the general domain.

3.1 n -DIMENSIONAL RECTANGLE WITH SINGLE VARIABLE.

For any non-negative function $f \in W_0^{1,p}(\Omega)$ with $\|f\|_{W_0^{1,p}} = 1$. In this subsection, we discuss $f(x_1, x_2, \dots, x_l, \dots, x_n)$ with x_l changed and other variables unchanged. We denote $\delta := (\frac{\phi^p(f)}{\mathcal{R}(f)})^{\frac{q}{p}}$. Given $I \subseteq \mathbb{R}^+$, let $L(I) := \int_{\{x \in \Omega | f(x) \in I\}} |f|^p dx$. We say I is W -dense if $L(I) \geq W$. For any $a \in \mathbb{R}^+$, we define

$$dist(a, I) := \inf_{b \in I} \frac{|a - b|}{b}. \quad (3.1)$$

The ε -neighborhood of a region I is the set $N_\varepsilon(I) := \{a \in \mathbb{R}^+ | dist(a, I) < \varepsilon\}$. If $N_\varepsilon(I_1) \cap N_\varepsilon(I_2) = \emptyset$, we say I_1, I_2 are ε -well separated.

Lemma 3.1. *Let I_1, \dots, I_{2k} be a set of W -dense and ε -well separated regions. Then, there are k disjointly supported functions f_1, \dots, f_k , each supported on the ε -neighborhood of one of the regions such that*

$$\mathcal{R}(f_i) \leq \frac{2^{p+1}\mathcal{R}(f)}{k\varepsilon^p W}, \forall 1 \leq i \leq k.$$

Proof. For any $1 \leq i \leq 2k$, we define the truncated function

$$f_i(x) := f(x) \max\{0, 1 - \frac{\text{dist}(f(x), I_i)}{\varepsilon}\}.$$

Then $\|f_i\|_p^p \geq L(I_i)$. Noting that the regions are ε -well separated, the functions are disjointly supported. By an averaging argument, there exist k functions f_1, \dots, f_k (after renaming) satisfy the following.

$$\int_{\Omega} |\nabla f_i|^p dx \leq \frac{1}{k} \sum_{j=1}^{2k} \int_{\Omega} |\nabla f_j|^p dx, \quad 1 \leq i \leq k.$$

By the construction of distance and $I_i \subset \mathbb{R}^1$, we know that

$$\int_{\Omega} |\nabla \max\{0, 1 - \frac{\text{dist}(f(x), I_i)}{\varepsilon}\}|^p |f(x)|^p dx \leq (\frac{1+\varepsilon}{\varepsilon})^p \int_{\Omega} |\nabla f(x)|^p dx.$$

Therefore

$$\int_{\Omega} |\nabla f_i(x)|^p dx \leq (1 + (\frac{1+\varepsilon}{\varepsilon})^p) \int_{\Omega} |\nabla f(x)|^p dx \leq (\frac{2}{\varepsilon})^p \int_{\Omega} |\nabla f|^p dx.$$

Thus, for $1 \leq i \leq k$,

$$\mathcal{R}(f_i) = \frac{\|\nabla f_i\|_p^p}{\|f_i\|_p^p} \leq \frac{\sum_{j=1}^{2k} \int_{\Omega} |\nabla f_j|^p dx}{k \min_{i \in [1, 2k]} \|f_i\|_p^p} \leq \frac{2(2^p \int_{\Omega} |\nabla f|^p dx)}{k\varepsilon^p W} = \frac{2^{p+1}\mathcal{R}(f)}{k\varepsilon^p W}.$$

□

Let $0 < \alpha < 1$ be a constant that will be fixed later. For $i \in \mathbb{Z}$, we define the interval $I_i := [\alpha^{i+1}, \alpha^i]$. We let $L_i := L(I_i)$. We partition each interval I_i into $12k$ subintervals of equal length.

$$I_{i,j} = [\alpha^i(1 - \frac{(j+1)(1-\alpha)}{12k}), \alpha^i(1 - \frac{j(1-\alpha)}{12k})], \quad \text{for } 0 \leq j \leq 12k.$$

So that $|I_{i,j}| = \frac{\alpha^i(1-\alpha)}{12k}$. Set $L_{i,j} = L(I_{i,j})$. We say a subinterval $I_{i,j}$ is heavy, if $L_{i,j} \geq \frac{c\delta L_{i-1}}{k}$, where $c > 0$ is a constant determined later. Otherwise we say it is light. We use \mathcal{H}_i to denote the set of heavy subintervals of I_i and \mathcal{L}_i for the set of light subintervals. Let $h_i := \#\mathcal{H}_i$ the number of heavy subintervals. If $h_i \geq 6k$, we say I_i is balanced, denoted by $I_i \in \mathcal{B}$.

Using Lemma 3.1, it is sufficient to find $2k, \frac{\delta}{k}$ -dense, $\frac{1}{k}$ well-separated regions R_1, R_2, \dots, R_{2k} , such that each regions are unions of heavy subintervals. We will use the following strategy: from each balanced interval we choose $2k$ separated heavy subintervals and include each of them in one of the regions. In order to keep that the regions are well separated, once we include $I_{i,j} \in \mathcal{H}_i$

into a region R we leave the two neighboring subintervals $I_{i,j-1}$ and $I_{i,j+1}$ unassigned, so as to separate R from the rest of the regions. In particular, for all $1 \leq a \leq 2k$ and all $I_i \in \mathcal{B}$, we include the $(3a-1)$ -th heavy subinterval of I_i in R_a . $R_a := \cup_{I_i \in \mathcal{B}} I_{i,a}$. Note that if an interval is balanced, then it has $6k$ heavy subintervals and we can include one heavy subinterval in each of the $2k$ regions. Moreover, by the construction of the distance function (3.1), the regions are $\frac{1-\alpha}{12k}$ -well separated. It remains to prove that these $2k$ regions are dense. Let

$$\Delta := \sum_{I_i \in \mathcal{B}} L_{i-1}.$$

Then, since each heavy subinterval $I_{i,j}$ has a mass of $\frac{c\delta L_{i-1}}{k}$, by the construction all regions are $\frac{c\delta\Delta}{k}$ -dense.

Therefore, we have the following lemma.

Lemma 3.2. *There are k disjoint supported functions f_1, \dots, f_k , such that for all $1 \leq i \leq k$, $\text{supp}(f_i) \subseteq \text{supp}(f)$, and*

$$\mathcal{R}(f_i) \leq \left(\frac{24k}{(1-\alpha)} \right)^p \frac{2\mathcal{R}(f)}{c\delta\Delta}, \quad \forall 1 \leq i \leq k.$$

Now we just need to lower bound Δ by an absolute constant.

Proposition 3.3. *For any interval $I_i \notin \mathcal{B}$,*

$$\mathcal{E}(I_i) \geq \frac{6(\alpha^{p(1+\frac{1}{q})}(1-\alpha))^p \phi^p(f) L_{i-1}}{(12)^p (c\delta)^{\frac{p}{q}}},$$

where $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. Claim: For any light interval $I_{i,j}$,

$$\mathcal{E}(I_{i,j}) \geq \frac{(\alpha^{p(1+\frac{1}{q})}(1-\alpha))^p \phi^p(f) L_{i-1}}{(c\delta)^{\frac{p}{q}} (12)^p k}.$$

Indeed, observe that

$$L_{i-1} = \int_{\Omega_f(I_{i-1})} |f(x)|^p dx \leq |\alpha^{i-1}|^p |\Omega_f(I_{i-1})| \leq |\alpha^{i-1}|^p |\Omega_f(\alpha^i)|.$$

Thus

$$\begin{aligned} |\Omega_f(I_{i,j})| &= \int_{\Omega_f(I_{i,j})} dx \leq \int_{\Omega_f(I_{i,j})} \frac{|f(x)|^p}{|\alpha^{i+1}|^p} dx = \frac{1}{(\alpha^{i+1})^p} \int_{\Omega_f(I_{i,j})} |f(x)|^p dx \\ &= \frac{1}{(\alpha^{i+1})^p} L_{i,j} \leq \frac{c\delta L_{i-1}}{k(\alpha^{i+1})^p} \leq \frac{c\delta |\alpha^{i-1}|^p |\Omega_f(I_{i-1})|}{k(\alpha^{i+1})^p} \leq \frac{c\delta |\Omega_f(\alpha^i)|}{k\alpha^{2p}}. \end{aligned}$$

where we use the assumption that $I_{i,j} \in \mathcal{L}_i$. Therefore, by Lemma 2.2,

$$\mathcal{E}(I_{i,j}) \geq \frac{(\phi(f)|\Omega_f(\alpha^i)||I_{i,j}|)^p}{|\Omega_f(I_{i,j})|^{\frac{p}{q}}} \geq \frac{(k\alpha^{2p})^{\frac{p}{q}} (\phi(f)|\Omega_f(\alpha^i)||I_{i,j}|)^p}{(c\delta |\Omega_f(\alpha^i)|)^{\frac{p}{q}}} = \frac{(k\alpha^{2p})^{\frac{p}{q}} |\Omega_f(\alpha^i)| (\phi(f)|I_{i,j}|)^p}{(c\delta)^{\frac{p}{q}}}.$$

Note that $|I_{i,j}| = \frac{\alpha^i(1-\alpha)}{12k}$, we have

$$\mathcal{E}(I_{i,j}) \geq \left(\frac{k\alpha^{2p}}{c\delta}\right)^{\frac{p}{q}} \frac{L_{i-1}}{|\alpha^{i-1}|^p} \left(\phi(f) \frac{\alpha^i(1-\alpha)}{12k}\right)^p = \frac{L_{i-1}(\phi(f)\alpha^{p(1+\frac{1}{q})}(1-\alpha))^p}{k(c\delta)^{\frac{p}{q}}(12)^p}.$$

Therefore, we get the Claim.

Now, since the subintervals are disjoint,

$$\mathcal{E}(I_i) \geq \sum_{I_{i,j} \in \mathcal{L}_i} \mathcal{E}(I_{i,j}) \geq (12k - h_i) \frac{L_{i-1}(\phi(f)\alpha^{p(1+\frac{1}{q})}(1-\alpha))^p}{k(c\delta)^{\frac{p}{q}}(12)^p} \geq \frac{6L_{i-1}(\phi(f)\alpha^{p(1+\frac{1}{q})}(1-\alpha))^p}{(c\delta)^{\frac{p}{q}}(12)^p},$$

where we used the assumption that I_i is not balanced and thus $h_i < 6k$. \square

Now, it is time to lower-bound Δ .

Note that $\|f\|_p = 1$.

$$\mathcal{R}(f) = \mathcal{E}(f) \geq \sum_{I_i \notin \mathcal{B}} \mathcal{E}(I_i) \geq \frac{6(\phi(f)\alpha^{p(1+\frac{1}{q})}(1-\alpha))^p}{(c\delta)^{\frac{p}{q}}(12)^p} \sum_{I_i \notin \mathcal{B}} L_{i-1}.$$

Therefore,

$$\sum_{I_i \notin \mathcal{B}} L_{i-1} \leq \frac{(c\delta)^{\frac{p}{q}}(12)^p \mathcal{R}(f)}{6(\phi(f)\alpha^{p(1+\frac{1}{q})}(1-\alpha))^p}.$$

Set $\alpha = \frac{1}{2}$ and $c^{\frac{p}{q}} := \frac{3(\alpha^{p(1+\frac{1}{q})}(1-\alpha))^p}{(12)^p}$. From the above inequality and the definition of δ , we get

$$\sum_{I_i \notin \mathcal{B}} L_{i-1} \leq \frac{1}{2}.$$

Note that $1 = \|f\|_p^p = \sum_{I_i \in \mathcal{B}} L_{i-1} + \sum_{I_i \notin \mathcal{B}} L_{i-1}$. Thus,

$$\Delta = \sum_{I_i \in \mathcal{B}} L_{i-1} \geq \frac{1}{2}.$$

Then, by Lemma 3.2 and the definition of δ , we get

$$\mathcal{R}(f_i) \leq \frac{2\mathcal{R}(f)}{c\delta\Delta} (48k)^p \leq Ck^p \left(\frac{\mathcal{R}(f)}{\phi(f)}\right)^q.$$

Therefore, we have

Theorem 3.4. *For any non-negative function $f \in W_0^{1,p}(\Omega)$, there are k disjoint supported functions f_1, \dots, f_k , such that for all $1 \leq i \leq k$, $\text{supp}(f_i) \subseteq \text{supp}(f)$, and*

$$\mathcal{R}(f_i) \leq Ck^p \left(\frac{\mathcal{R}(f)}{\phi(f)}\right)^q, \quad \forall 1 \leq i \leq k,$$

where C depends only on p .

Remark 3.5. The above arguments can also be used in general dimension $n > 1$ without any modification.

3.2 GENERAL n -DIMENSIONAL CASES.

Using arguments as in above subsection, we will first discuss the case of n -dimensional rectangle $\Omega = (a_1, b_1) \times (a_2, b_2) \times \cdots (a_n, b_n) \subset \mathbb{R}^n$ with multi-variables changed. Then, we deal with the general domain by comparing the volume of Ω and the inscribed rectangle.

When Ω is an n -dimensional rectangle $\Omega = (a_1, b_1) \times (a_2, b_2) \times \cdots (a_n, b_n) \subset \mathbb{R}^n$. we get a similar theorem as Theorem 3.4.

Theorem 3.6. *For the first eigenfunction $f \in W_0^{1,p}(\Omega)$, there are k^n disjoint supported functions $f_{i,j}(x)$, such that for all $1 \leq i \leq k, 1 \leq j \leq n$, $\text{supp}(f_{i,j}(x)) \subseteq \text{supp}(f(x))$, and*

$$\mathcal{R}(f_{i,j}(x)) \leq Ck^p \left(\frac{\mathcal{R}(f)}{\phi(f)} \right)^q, \quad \forall 1 \leq i \leq k, 1 \leq j \leq n,$$

where C depends only on n, p .

Proof. In the proof of Lemma 3.1, we set $\theta_{i,j}(x) = \max\{0, 1 - \frac{\text{dist}(f(x_1, \dots, x_j, \dots, x_n), I_i)}{\varepsilon}\}$, where $1 \leq i \leq k, 1 \leq j \leq n$. For each variable, discussing as in subsection 3.1, we get k^n support separated functions $f_{i,j}(x) = f(x)\theta_{i,j}(x)$, where $1 \leq i \leq k, 1 \leq j \leq n$. By the construction, $\text{supp}(f_{i,j}(x)) \subseteq \text{supp}(f(x))$ and

$$\mathcal{R}(f_{i,j}(x)) \leq Ck^p \left(\frac{\mathcal{R}(f)}{\phi(f)} \right)^q, \quad \forall 1 \leq i \leq k, 1 \leq j \leq n,$$

where C depends only on n, p . Therefore, we get the theorem. \square

Finally, the case of a general bounded domain Ω with comparable inscribed n -dimensional rectangle $R \subset \Omega$, can be proved by comparison. More precisely, for the first eigenfunction f of R , by Theorem 3.6, we can find k^n functions $f_{i,j}(x)$. Noting that Lemma 2.1, we have k^n subset $S_{i,j} \subset R \subset \Omega$, such that

$$\phi(S_{i,j}) \leq p(\mathcal{R}(f_{i,j}(x)))^{\frac{1}{p}} \leq Ck \left(\frac{\mathcal{R}(f)}{\phi(f)} \right)^{\frac{q}{p}}.$$

Redefining the subscript, by the definition of $h_k(\Omega)$, we have

$$h_k(\Omega) \leq h_k(R) \leq Ck^{\frac{1}{n}} \left(\frac{\mathcal{R}(f)}{\phi(f)} \right)^{\frac{q}{p}} \leq Ck^{\frac{1}{n}} \left(\frac{\lambda_1(p, R)}{h_1(R)} \right)^{\frac{q}{p}}.$$

Therefore, we get theorem 1.2.

When Ω is convex, (1.7) and (1.9) substituted into the above inequalities, we get

$$h_k(\Omega) \leq Ck^{\frac{1}{n}} \left(\frac{1}{|\Omega|} \right)^{\frac{(p-1)q}{np}} = C \left(\frac{k}{|\Omega|} \right)^{\frac{1}{n}}.$$

Combining (1.7) with the above inequality, we obtain Theorem 1.5.

Again, using (1.9), there exist C , such that

$$h_k^p(\Omega) \leq C\lambda_k(p, \Omega).$$

Thus we prove corollary 1.6.

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